[nex47] Ultracold neutrons in an ideal Steyerl bottle

Consider a container whose walls are perfect mirrors for ultracold neutrons. At time $t = 0$ the bottle is known to contain exactly n_0 neutrons. The decay rate of a neutron is K.
(a) Set up the master equation,

$$\frac{\partial}{\partial t} P(n,t) = (n+1)KP(n+1,t) - KnP(n,t),$$

for the probability distribution $P(n,t)$, and derive the PDE,

$$\frac{\partial G}{\partial t} + K(z-1)\frac{\partial G}{\partial z} = 0,$$

for the generating function $G(z,t) \doteq \sum_n z^n P(n,t)$.
(b) Solve the PDE by the method of characteristics (see e.g. [nex112]) to obtain

$$G(z,t) = \left[(z-1)e^{-Kt} + 1\right]^{n_0}.$$

(c) Infer therefrom the probability distribution

$$P(n,t) = \frac{n_0!}{n!(n_0-n)!} \frac{\left(1-e^{-Kt}\right)^{n_0}}{\left(e^{Kt}-1\right)^{n}}.$$

(d) Determine (via derivatives of the generating function) the average number $\langle n(t)\rangle$ of remaining neutrons and the variance $\langle\langle n^2(t)\rangle\rangle$ thereof.
(e) Design a contour plot of $P(n,t)$ for $0 < n < n_0 = 20$ and $0 < Kt < 3$. Design a line graph of $P(n,t)$ for $0 < Kt < 5$ for fixed $n = 0, 1, 2, 5, 10, 15, 18, 19$. Interpret these graphs.
(f) Derive equations of motion for $\langle n\rangle$ and $\langle n(n-1)\rangle$ directly from the master equation and solve them to reproduce the solutions obtained in (d).

Solution: