[nex115] Modified linear birth rate III: generating function

Consider a linear birth-death process with a modified birth rate to be used in the master equation:

$$W(m|n) = \lambda(n+1)\delta_{m,n+1} + \mu n\delta_{m,n-1}, \quad \lambda < \mu.$$

In [nex44] we had shown that the original linear birth rate $T_+(n) = n\lambda$ leads to the extinction of the population. In [nex113] we showed that the modified linear birth rate $T_+(n) = (n+1)\lambda$ leads to a nonvanishing stationary distribution $P_s(n)$: the Pascal distribution.
(a) Construct the PDE,

$$\frac{\partial G}{\partial t} - (\lambda z - \mu)(z-1)\frac{\partial G}{\partial z} = \lambda(z-1)G,$$

for the generating function $G(z,t) = \sum_n z^n P(n,t)$ and to solve that PDE for the case of zero initial population by the method of characteristics (see e.g. [nex112]). The result reads

$$G(z,t) = \frac{\lambda - \mu}{\lambda z - \mu - \lambda(z-1)e^{(\lambda-\mu)t}}.$$

(b) Calculate mean $\langle\langle n(t)\rangle\rangle$ and variance $\langle\langle n^2(t)\rangle\rangle$ from derivative of the $G(z,t)$ for comparison with the results obtained in [nex114] via a different route.
(c) Calculate the stationary distribution $P_s(n)$ from the asymptotic generating function $G(z,\infty)$.

Solution: