Relaxation Function and Spectral Density [nln84]

Relaxation function via Laplace transform:

$$c_k(z) \doteq \int_0^\infty dt\, e^{-zt} C_k(t). \tag{1}$$

Coupled ODEs for $C_k(t)$ become coupled algebraic equations for $c_k(z)$:[1]

$$zc_k(z) - \delta_{k,0} = c_{k-1}(z) - \Delta_{k+1} c_{k+1}(z), \quad k = 0, 1, 2, \ldots \tag{2}$$

Condition: $c_{-1}(z) \equiv 0$.

Recursive construction of continued fraction representation for $c_0(z)$:

$$k = 0: \; zc_0(z) - 1 = -\Delta_1 c_1(z) \quad \Rightarrow \; c_0(z) = \cfrac{1}{z + \Delta_1 \cfrac{c_1(z)}{c_0(z)}},$$

$$k = 1: \; zc_1(z) = c_0(z) - \Delta_2 c_2(z) \quad \Rightarrow \; \frac{c_1(z)}{c_0(z)} = \cfrac{1}{z + \Delta_2 \cfrac{c_2(z)}{c_1(z)}},$$

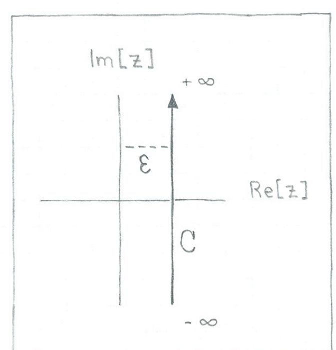
$$\Rightarrow \; c_0(z) = \cfrac{1}{z + \cfrac{\Delta_1}{z + \cfrac{\Delta_2}{z + \cdots}}} \tag{3}$$

Spectral density $\Phi(\omega)$ from relaxation function $c_0(z)$: combine Fourier transform with inverse Laplace transform.

$$\Phi_0(\omega) \doteq \int_{-\infty}^{+\infty} dt\, e^{\imath\omega t} C_0(t),$$

$$C_0(t) = \frac{1}{2\pi\imath} \int_{\mathcal{C}} dz\, e^{zt} c_0(z)$$

$$\Rightarrow \; \Phi_0(\omega) = 2 \lim_{\epsilon \to 0} \mathrm{Re}[c_0(\epsilon - \imath\omega)]. \tag{4}$$



[1] Use $\int_0^\infty dt\, e^{-zt} \dot{C}_k(t) = \left[e^{-zt} C_k(t)\right]_0^\infty - \int_0^\infty dt (-z) e^{-zt} C_k(t) = zc_k(z) - C_k(0)$.