Characteristic Function [nln47]

Fourier transform: $\Phi_X(k) \doteq \left\langle e^{ikx} \right\rangle = \int_{-\infty}^{+\infty} dx\, e^{ikx} P_X(x).$

Attributes: $\Phi_X(0) = 1, \quad |\Phi_X(k)| \leq 1.$

Moment generating function:

$$\Phi_X(k) = \int_{-\infty}^{+\infty} dx\, P_X(x) \left[\sum_{n=0}^{\infty} \frac{(ik)^n}{n!} x^n \right] = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle X^n \rangle$$

$$\Rightarrow \ \langle X^n \rangle \doteq \int_{-\infty}^{+\infty} dx\, x^n P_X(x) = (-i)^n \frac{d^n}{dk^n} \Phi_X(k) \bigg|_{k=0}.$$

Cumulant generating function:

$$\ln \Phi_X(k) \doteq \sum_{n=1}^{\infty} \frac{(ik)^n}{n!} \langle\langle X^n \rangle\rangle \quad \Rightarrow \ \langle\langle X^n \rangle\rangle = (-i)^n \frac{d^n}{dk^n} \ln \Phi_X(k) \bigg|_{k=0}.$$

Cumulants in terms of moments (with $\Delta X \doteq X - \langle X \rangle$): [nex126]

- $\langle\langle X \rangle\rangle = \langle X \rangle$
- $\langle\langle X^2 \rangle\rangle = \langle X^2 \rangle - \langle X \rangle^2 = \langle (\Delta X)^2 \rangle$
- $\langle\langle X^3 \rangle\rangle = \langle (\Delta X)^3 \rangle$
- $\langle\langle X^4 \rangle\rangle = \langle (\Delta X)^4 \rangle - 3\langle (\Delta X)^2 \rangle^2$

Theorem of Marcienkiewicz:
$\ln \Phi_X(k)$ can only be a polynomial if the degree is $n \leq 2$.

- $n = 1$: $\ln \Phi_X(k) = ika \quad \Rightarrow \ P_X(x) = \delta(x - a)$
- $n = 2$: $\ln \Phi_X(k) = ika - \frac{1}{2} bk^2 \quad \Rightarrow \ P_X(x) = \frac{1}{\sqrt{2\pi b}} \exp\left(-\frac{(x-a)^2}{2b} \right)$

Consequence: any probability distribution has either one, two, or infinitely many non-vanishing cumulants.