[pex38] Percolation on Bethe lattice

The Bethe lattice with coordination number z is a tree-like structure emanating from a central node out to infinity. Each node bonds with one node down the tree and $z-1$ nodes up the tree. The figure shows three generations of bonds for $z=3$. The number of nodes or bonds in generation k is $N_k = z(z-1)^{k-1}$. Now we connect nodes randomly with probability f, meaning that any bond is active with probability f and inactive with probability $1-f$. Note that the number of nodes out to any generation of is equal to the number of bonds.

(a) Show and reason that the threshold fraction of active bonds that produce an infinite cluster with non-vanishing probability is

$$f_c = \frac{1}{z-1}.$$

(b) If P is the probability that a node is connected to infinity and Q the probability that a node is not connected to infinity via a specific neighbor then show that the following relation holds:

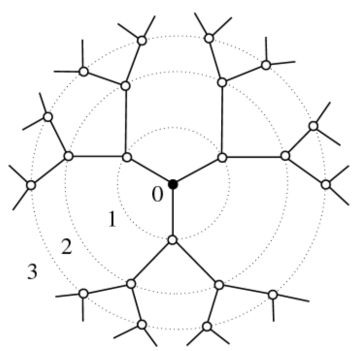
$$P = f - fQ^z$$

(c) Next show that Q is the solution of the polynomial equation,

$$Q = 1 - f + fQ^{z-1}.$$

(d) The gel fraction is equal to the fraction of active bonds that are part of the infinite network. That fraction is P/f. Plot the gel fraction P/f vs f for $0 < f < 1$ and $z = 3, 4, 5$ as three curves in the same graph.

[adapted from Jones 2002]



[image from Wikipedia]

Solution: