[pex34] Polymer with energetically favored internal rotation angles II

Consider an ideal polymer chain with fixed valence angle $\cos_{i,i+1} = \gamma$ and an internal rotation angle $\phi_{i,i+1} \doteq \phi_i$ subject to a potential $U(\phi_i)$ with symmetry $U(-\phi_i) = U(\phi_i)$. Here we use the result from [pex33] for the angle $\theta_{i,i+k}$ between links i and k to calculate the thermal average $\langle\theta_{i,i+k}\rangle$ for use in the expression from [pex30] for the mean-square end-to-end distance $\langle R^2\rangle$.
(a) Provide a chain of reasoning that leads to the following result:

$$\langle\theta_{i,i+k}\rangle = \left[\langle\mathbf{T}\rangle^k\right]_{11}, \quad \langle\mathbf{T}\rangle = \begin{pmatrix} \cos\gamma & \sin\gamma\langle\cos\phi\rangle & 0 \\ \sin\gamma & -\cos\gamma\langle\cos\phi\rangle & 0 \\ 0 & 0 & -\langle\cos\phi\rangle \end{pmatrix}.$$

(b) Provide a chain of reasoning that leads to the following expression for the mean-square end-to-end distance:

$$\langle R^2\rangle \overset{N\gg 1}{\rightsquigarrow} Na^2 \left[\frac{\mathbf{E}+\langle\mathbf{T}\rangle}{\mathbf{E}-\langle\mathbf{T}\rangle}\right]_{11}, \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Evaluate this expression via the solution of the matrix equation $\mathbf{X}\cdot(\mathbf{E}-\langle\mathbf{T}\rangle) = \mathbf{E}+\langle\mathbf{T}\rangle$.

[adapted from Grosberg and Khokhlov 1994]

Solution: