[pex30] Ideal polymer chain with fixed valence angle I

Consider an ideal polymer chain of N links of length a with fixed valence angle γ between successive links. The internal rotational angle is assumed to be unconstrained and not subject to a potential: $U(\varphi) = 0$. Calculate the mean-square end-to-end distance from the expression,

$$\langle R^2 \rangle = \left\langle \left(\sum_{i=1}^{N} \vec{u}_i \right)^2 \right\rangle = Na^2 + 2a^2 \sum_{i=1}^{N} \sum_{k=1}^{N-i} \langle \cos\theta_{i,i+k} \rangle,$$

where the $\vec{u}_i$ are bond vectors of length a. For given uniform valence angle γ use the multiplicativity property $\langle \cos\theta_{i,i+k} \rangle = (\cos\gamma)^k$ for the angle between further-neighbor bond vectors. Show that the result can be written in the form,

$$\langle R^2 \rangle = Na^2 \left[\frac{1+\cos\gamma}{1-\cos\gamma} - N^{-1} \frac{2\cos\gamma}{1-\cos\gamma} \frac{1-(\cos\gamma)^N}{1-\cos\gamma} \right].$$

In very long polymers only the first term survives. Explain this solution for $N = 1, 2, 3$ in geometric terms.

[adapted from Grosberg and Khokhlov 1994]

Solution: