[pex20] Solution of linearized Cahn-Hilliard equation

The unmixing process of two liquids from unstable macrostates (spinodal decomposition) is initiated by small local fluctuations in concentration. A quantitative analysis of this process leads, under certain assumptions, to a nonlinear partial differential equation for the concentration $\phi(\vec{r})$ of one or the other liquid: the Cahn-Hilliard equation. One characteristic attribute of the morphological patterns emerging during the unmixing process can be found already in the linearized Cahn-Hilliard equation for a single space coordinate,

$$\frac{\partial \phi}{\partial t} = -Mc\frac{\partial^2 \phi}{\partial x^2} - 2M\kappa\frac{\partial^4 \phi}{\partial x^4},$$

where $M > 0$ is a transport coefficient, $\kappa > 0$ is an energy coefficient associated with inhomogeneities, and $c > 0$ is a measure for the instability of the mixed macrostate.
(a) Show that the concentration profile with characteristic wave number q,

$$\phi(x,t) = \phi_0 + a\cos(qx)\exp\big(R(q)t\big), \quad R(q) \doteq M(cq^2 - 2\kappa q^4)$$

is an exact solution of the linearized Cahn-Hilliard equation.
(b) Visualize the amplification factor $R(q)$ in a scaled plot of universal shape.
(c) Morphological patterns with different wave numbers q are amplified or suppressed at different rates. Find the wave number q_0 for which morphological patterns are amplified most.

[adapted from Jones 2002]

Solution: