[pex11] Polymer gel compressed uniaxially

Consider a rectangular slab of polymer gel with volume fraction ϕ_0 when dry and $\phi_1 < \phi_0$ when immersed and swelled in solvent. Now this swelled slab is being compressed uniaxially (in z-direction), causing elongations in x-direction and y-direction. We expect the volume of the slab to be reduced somewhat in the process The associated increase in volume fraction implies that some amount of solvent will be squeezed out of the gel.
We construct the free-energy-density of the compressed polymer gel using the expression developed in [pln65] with $\lambda_1 = \lambda_2 = (\phi_0/\phi_1)^{1/3}\lambda_x$ and $\lambda_3 = (\phi_0/\phi_1)^{1/3}\lambda_z$ to account for the change in reference state from dry gel to immersed gel:

$$f_{\text{gel}} = \frac{1}{2}G_0\left[\left(\frac{\phi_0}{\phi_1}\right)^{2/3}\left(\lambda_z^2 + 2\lambda_x^2\right) - 3\right] + \frac{\phi_0}{\phi}\, f_{\text{sol}}(\phi).$$

The uniaxial compression will be described by $\lambda_x > 1$ and $\lambda_z < 1$. In order to find the equilibrium polymeric volume fraction ϕ of the uniaxially compressed state as a function of length ratio λ_z in the direction of the compression force we take two steps:
(i) Express λ_x as a function of λ_z and ϕ using conservation of polymeric volume.
(ii) Find a relation between ϕ and λ_z from the extremum condition $\partial f_{\text{gel}}/\partial\phi = 0$. Confirm that compression, $\lambda_z < 1$, leads to an increase in ϕ, implying a smaller volume of the slab of polymeric gel. In that relation, identify the osmotic pressure $\pi_{\text{sol}}(\phi)$ as defined in [pex13], which facilitates the interpretation.

[adapted from Doi 2013]

Solution: