[pex1] Attributes of nematic order parameter

Consider the nematic order parameter proposed in [pln80]:

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2}\langle 3\cos^2\theta - 1\rangle = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, g(\theta,\phi)\big(3u_\alpha u_\beta - \delta_{\alpha\beta}\big),$$

where

$$u_x = \sin\theta\cos\phi, \quad u_y = \sin\theta\sin\phi, \quad u_z = \cos\theta.$$

(a) Show that all elements of $\mathcal{N}_{\alpha\beta}$ vanish in the isotropic phase, where the orientation function is uniform:

$$g(\theta,\phi)_{\text{iso}} = \frac{1}{4\pi}.$$

(b) Show that the tensor $\mathcal{N}_{\alpha\beta}$ is diagonal if the nematic ordering is perfect and aligned with one axis, for example the x-axis:

$$g(\theta,\phi)_{\text{nem}} = \delta(\phi)\delta(\theta - \pi/2).$$

Calculate the diagonal elements $\mathcal{N}_{\alpha\alpha}$, $\alpha = x, y, z$ for this case.

Solution: