

[mex19] **Hopf bifurcation**

A simple Hopf bifurcation generates a limit cycle from a point attractor upon variation of some parameter in the equations of motion of a dynamical system. Consider the dynamical system specified (in polar coordinates) by

$$\dot{r} = -\Gamma r - r^3, \quad \dot{\theta} = \omega,$$

where Γ and ω are constants.

- (a) Find the exact solution $r(t), \theta(t)$ for initial conditions $r(0) = r_0, \theta(0) = 0$.
- (b) Identify the circular periodic trajectory for $\Gamma < 0$, which plays the role of a limit cycle, and determine its radius.
- (c) Determine the nature of the fixed point at $r = 0$ for $\Gamma > 0$ and $\Gamma < 0$.
- (d) Sketch the phase portrait for $\Gamma > 0$ and $\Gamma < 0$.