Driven Harmonic Oscillator III [pmln1]

No damping and arbitrary driving force.

Equation of motion: $m\ddot{x} + \omega_0^2 x = F(t)$,

Complex variable: $\xi(t) \doteq \dot{x}(t) + \imath\omega_0 x(t) \quad \Rightarrow \dot{\xi}(t) - \imath\omega_0\xi(t) = F(t)/m$.

Ansatz: $\xi(t) = B(t)e^{\imath\omega_0 t} \quad \Rightarrow \dot{B}(t) = \frac{1}{m}F(t)e^{-\imath\omega_0 t}$.

Solution: $x(t) = \frac{1}{\omega_0}\Im[\xi(t)], \quad \xi(t) = e^{\imath\omega_0 t}\left[\frac{1}{m}\int_0^t dt' F(t')e^{-\imath\omega_0 t'} + \xi_0\right]$.

Instantaneous energy at time t or total energy absorbed at time t if oscillator is initially at equilibrium ($x_0 = \dot{x}_0 = 0$):

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}m|\xi(t)|^2 = \frac{1}{2m}\left|\int_0^t dt' F(t')e^{-\imath\omega_0 t'}\right|^2.$$

Constant force switched on: $F(t) = F_0\Theta(t)$.

$$x(t) = \frac{F_0}{m\omega_0^2}\Big(1 - \cos\omega_0 t\Big), \quad E(t) = \frac{F_0^2}{m\omega_0^2}\Big(1 - \cos\omega_0 t\Big).$$

Force performs positive and negative work in alternation.

Fading force switched on: $F(t) = F_0 e^{-\alpha t}\Theta(t)$.

$$x(t) = \frac{F_0}{m(\omega_0^2 - \alpha^2)}\left[e^{-\alpha t} - \cos\omega_0 t + \frac{\alpha}{\omega_0}\sin\omega_0 t\right].$$

$$E(t) = \frac{F_0^2}{2m(\omega_0^2 + \alpha^2)}\Big[1 + e^{-2\alpha t} - 2e^{-\alpha t}\cos\omega_0 t\Big].$$

Constant force switched on and then off: $F(t) = F_0\Theta(t)\Theta(T - t)$.

$$x(t) = \frac{F_0}{m\omega_0^2}\Big[\cos\omega_0(t - T) - \cos\omega_0 t\Big] \qquad (t \geq T).$$

$$E(t) = \frac{2F_0^2}{m\omega_0^2}\sin^2\frac{\omega_0 T}{2} = \text{const} \qquad (t \geq T).$$