Fundamental Equations of Microfluidics III [pln87]

Energy flux:

First law of thermodynamics:

$$d\epsilon = T ds - p d\big(\rho^{-1}\big) = T ds + \frac{p}{\rho^2} d\rho,$$

- ρ: mass density (mass per volume),
- ϵ: energy per mass,
- $\rho\epsilon$: energy density (energy per volume),
- s: entropy per mass,
- ρs: entropy density (energy per volume).

Energy inside Ω: $E(\Omega,t) = \int_\Omega d\mathbf{r}\Big[\underbrace{\tfrac{1}{2}\rho(\mathbf{r},t)v^2(\mathbf{r},t)}_{\text{kinetic}} + \underbrace{\rho(\mathbf{r},t)\epsilon(\mathbf{r},t)}_{\text{internal}}\Big].$

Agents of energy change:

- convection,
- stress forces (pressure, viscous),
- heat conduction.

Rate of energy change (power conversion):

$$\frac{\partial}{\partial t}E(\Omega,t) = \frac{\partial}{\partial t}E(\Omega,t)_{\text{conv}} + \frac{\partial}{\partial t}E(\Omega,t)_{\text{stre}} + \frac{\partial}{\partial t}E(\Omega,t)_{\text{cond}}.$$

Energy convection:[1]

$$\frac{\partial}{\partial t}E(\Omega,t)_{\text{conv}} = -\int_{\partial\Omega} da\, \mathbf{n}\cdot\mathbf{J}_\epsilon = -\int_{\partial\Omega} da\, n_j v_j \big[\tfrac{1}{2}\rho v^2 + \rho\epsilon\big],$$

- energy flux density: $\mathbf{J}_\epsilon = \big[\tfrac{1}{2}\rho v^2 + \rho\epsilon\big].$

Work of stress forces:

$$\frac{\partial}{\partial t}E(\Omega,t)_{\text{stre}} = \int_{\partial\Omega} da\, v_k \sigma_{kj} n_j = \int_{\partial\Omega} da \big[-p\delta_{jk} + \sigma'_{jk}\big] v_k.$$

[1]Summation over repeated indices implied.

Heat conduction:

$$\frac{\partial}{\partial t}E(\Omega,t)_{\rm cond} = -\int_{\partial\Omega} da\,\mathbf{n}\cdot\mathbf{J}_{\rm heat} = \int_{\partial\Omega} da\,n_j\,\kappa\frac{\partial T}{\partial r_j},$$

- Fourier's law: $\mathbf{J}_{\rm heat} = -\kappa\nabla T$,
  - $T(\mathbf{r})$: temperature field,
  - $\mathbf{J}_{\rm heat}(\mathbf{r})$: heat flux density,
  - κ: thermal conductivity,

Conversion of surface integrals into volume integrals via Gauss's theorem leads to a raw version of the *heat transfer equation*:

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho v^2 + \rho\epsilon\right] = -\frac{\partial}{\partial r_j}\left(\left[\tfrac{1}{2}\rho v^2 + \rho\epsilon + p\right]v_j - \sigma'_{jk}v_k - \kappa\frac{\partial T}{\partial r_j}\right)$$
$$= -\nabla\cdot\underbrace{\left(\left[\tfrac{1}{2}\rho v^2 + \rho\epsilon + p\right]\mathbf{v} - \sigma'\cdot\mathbf{v} - \kappa\nabla T\right)}_{\text{energy flux density}}.$$

Transformation of the left-hand side:

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho v^2 + \rho\epsilon\right] = \left(\tfrac{1}{2}v^2 + \epsilon\right)\underbrace{\frac{\partial\rho}{\partial t}}_{\rm (a)} + \underbrace{\rho v_j\frac{\partial v_j}{\partial t}}_{\rm (b)} + \underbrace{\rho\frac{\partial\epsilon}{\partial t}}_{\rm (c)}.$$

(a) Use continuity equation: $\dfrac{\partial\rho}{\partial t} = -\dfrac{\partial(\rho v_j)}{\partial r_j}$.

(b) Use equation of motion [pln86],

$$\rho v_j\frac{\partial v_j}{\partial t} = \underbrace{-\rho v_j v_k\frac{\partial v_j}{\partial r_k}}_{-\rho v_k\frac{\partial}{\partial r_k}\left(\frac{1}{2}v^2\right)} + v_j\frac{\partial\sigma'_{jk}}{\partial r_k} - \underbrace{v_j\frac{\partial p}{\partial r_j}}_{\#},$$

and first law,

$$d\left(\epsilon + \frac{p}{\rho}\right) = \Big[Tds - pd\big(\rho^{-1}\big)\Big] + \Big[pd\big(\rho^{-1}\big) + \rho^{-1}dp\Big] = Tds + \rho^{-1}dp$$

$$\Rightarrow\ dp = \rho d\left(\epsilon + \frac{p}{\rho}\right) - T\rho ds \quad\Rightarrow\ \# = \rho v_j\frac{\partial}{\partial r_j}\left(\epsilon + \frac{p}{\rho}\right) - \rho T v_j\frac{\partial s}{\partial r_j}.$$

(c) Use first law and continuity equation [pln85]:

$$\rho\frac{\partial\epsilon}{\partial t} = \rho T\frac{\partial s}{\partial t} + \frac{p}{\rho}\frac{\partial\rho}{\partial t} = \rho T\frac{\partial s}{\partial t} - \frac{p}{\rho}\frac{\partial(\rho v_j)}{\partial r_j}.$$

Substitution of (a), (b), and (c):

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho v^2 + \rho\epsilon\right] = -\frac{\partial}{\partial r_j}\left[\left(\tfrac{1}{2}\rho v^2 + \rho\epsilon + p\right)v_j\right] + v_j\frac{\partial \sigma'_{jk}}{\partial r_k} + \rho T\left[\frac{\partial s}{\partial t} + v_j\frac{\partial s}{\partial r_j}\right].$$

Consolidated version of *heat transfer equation* (in two distinct notations):

$$\rho T\left[\frac{\partial s}{\partial t} + v_j\frac{\partial s}{\partial r_j}\right] = \sigma'_{jk}\frac{\partial v_j}{\partial r_k} + \frac{\partial}{\partial r_j}\left(\kappa\frac{\partial T}{\partial r_j}\right),$$

$$\underbrace{\rho T\left[\frac{\partial s}{\partial t} + \left(\mathbf{v}\cdot\nabla s\right)\right]}_{ds/dt} = \underbrace{\sigma' : \nabla\mathbf{v}}_{(i)} + \underbrace{\nabla\cdot\left(\kappa\nabla T\right)}_{(ii)}.$$

Sources of heat transfer: (i) viscous friction, (ii) heat conduction.

In microfluidics flow velocities are sufficiently small to make pressure variations due to motion negligible.

Consequences: $\dfrac{\partial s}{\partial t} = \left(\dfrac{\partial s}{\partial T}\right)_p\dfrac{\partial T}{\partial t}, \quad \nabla s = \left(\dfrac{\partial s}{\partial T}\right)_p\nabla T, \quad \left(\dfrac{\partial s}{\partial T}\right)_p = \dfrac{c_p}{T}.$

Simplified *heat transfer equation*:

$$\rho c_p\left[\frac{\partial T}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)T\right] = \sigma' : \nabla\mathbf{v} + \nabla\cdot\left(\kappa\nabla T\right).$$

If temperature variations are small then ρ, κ, η, c_p can be treated as constants. The associated *heat transfer equation* is further simplified into [pex63]:

$$\frac{\partial T}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)T = D_{\mathrm{th}}\nabla^2 T + \frac{\eta}{2\rho c_p}\left(\frac{\partial v_j}{\partial r_k} + \frac{\partial v_k}{\partial r_j}\right)^2,$$

- thermal diffusivity: $D_{\mathrm{th}} \doteq \dfrac{\kappa}{\rho c_p}$,
- Fourier's law: $\dfrac{\partial T}{\partial t} = D_{\mathrm{th}}\nabla^2 T$ (for fluid at rest).

[extracted from Bruus 2008]