Rubber Elasticity [pln64]

Consider a dry chemical gel of cross-linked polymers (see [psl11]).

Model of deformation free-energy density:

$$f(\mathbf{E}) = \frac{1}{2} n_\mathrm{c} k_\mathrm{B} T \left[\sum_{\alpha\beta} E_{\alpha\beta}^2 - 3 \right].$$

- n_c: number of segments between cross links per unit volume,
- $\mathbf{E}$: deformation gradient tensor (see pln63]).

Consider two principal types of deformation:

1. **Shear deformation:**

$$\mathbf{E} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \; f(\gamma) = \frac{1}{2} n_\mathrm{c} k_\mathrm{B} T \gamma^2,$$

shear strain: $e = \gamma \ll 1$,

shear stress: $\sigma \doteq Ge = \dfrac{\partial f}{\partial \gamma} = n_\mathrm{c} k_\mathrm{B} T \gamma$,

shear modulus: $G = n_\mathrm{c} k_\mathrm{B} T$.

2. **Tensile deformation** (with incompressibility constraint):

$$\mathbf{E} = \begin{pmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \Rightarrow \; f(\lambda) = \frac{1}{2} G \left(\lambda^2 + \frac{2}{\lambda} - 3 \right),$$

tensile strain: $\epsilon \ll 1$ from $\lambda = 1 + \epsilon$,

tensile stress:[1] $\sigma \doteq Y\epsilon = \lambda \dfrac{\partial f}{\partial \lambda} = G \left(\lambda^2 - \dfrac{1}{\lambda} \right) \;\rightsquigarrow\; 3G\epsilon$,

Young modulus: $Y = 3G = 3 n_\mathrm{c} k_\mathrm{B} T$.

[gleaned from Doi 2013]

[1] The factor λ arises due to the shrinking of the cross section upon elongation.