Elastic Energy of Deformation [pln63]

Within a continuum description of elastic material calculate

- the deformation free-energy density $f(\mathbf{r})$,
- the deformation free energy $F = \int d^3r f(\mathbf{r})$.

We consider three successive stages of generality.

1. **Orthogonal deformation**
   Pure extension or compression in three orthogonal directions, expressed locally as follows:

$$r'_x = \lambda_1 r_x, \quad r'_y = \lambda_2 r_y, \quad r'_z = \lambda_3 r_z.$$

$$\Rightarrow \; f = f(\lambda_1, \lambda_2, \lambda_3), \quad \text{independent of position } \mathbf{r}.$$

2. **Uniform deformation**
   Linear transformation, expressed locally by deformation gradient tensor $\mathbf{E}$ as follows:

$$\mathbf{r}' = \mathbf{E} \cdot \mathbf{r} \quad \text{or} \quad r'_\alpha = \sum_\beta E_{\alpha\beta} r_\beta, \quad \text{where} \quad E_{\alpha\beta} = \frac{\partial r'_\alpha}{\partial r_\beta}.$$

   Convenient representation:[1] $\mathbf{E} = \mathbf{Q} \cdot \mathbf{L}$, where

   - $\sum_\gamma Q_{\alpha\gamma} Q_{\beta\gamma} = \delta_{\alpha\beta}$ (orthogonal tensor),
   - $L_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$ (diagonal tensor),
   - $\det(\mathbf{E}) = \lambda_1 \lambda_2 \lambda_3$ (determinant of $\mathbf{E}$),
   - $\sum_{\alpha\beta} E^2_{\alpha\beta} = \sum_\alpha B_{\alpha\alpha} = \sum_\alpha \lambda^2_\alpha$ (trace of $\mathbf{E} \cdot \mathbf{E}^t$).

$$\Rightarrow \; f = f(\lambda_1, \lambda_2, \lambda_3), \quad \text{still independent of position } \mathbf{r}.$$

3. **General deformation**
   Local deformation assumed to be uniform.

$$\Rightarrow \; F = \int d^3r f\big(\mathbf{E}(\mathbf{r})\big).$$

   Equilibrium deformation from minimization of F for given boundary conditions.

[1] Symmetric tensor $\mathbf{B} \doteq \mathbf{E} \cdot \mathbf{E}^t$ is diagonalized by orthogonal tensor $\mathbf{Q}$: $\mathbf{Q}^t \cdot \mathbf{B} \cdot \mathbf{Q} = \mathbf{L}^2$. It follows that $\mathbf{E} \cdot \mathbf{E}^t = \mathbf{B} = (\mathbf{Q} \cdot \mathbf{L}) \cdot (\mathbf{L} \cdot \mathbf{Q}^t)$.

General expression for deformation energy density:

$$f(\mathbf{E}) \propto \mathrm{Tr}\big(\mathbf{E}\cdot\mathbf{E}^t - 3\big).$$

Consider two principal types of deformation:

- shear deformation:

$$\mathbf{E} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma > 0.$$

- uniform swelling:[2]

$$\mathbf{E} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_i > 1.$$

- tensile deformation:

$$\mathbf{E} = \begin{pmatrix} \lambda_\perp^{-1/2} & 0 & 0 \\ 0 & \lambda_\perp^{-1/2} & 0 \\ 0 & 0 & \lambda_\parallel \end{pmatrix},$$

  where $\lambda_\perp = \lambda_\parallel$ means incompressibility.

[in part from Doi 2013]

[2]For later applications to gels.