

Binomial, Poisson, and Gaussian Distributions [nln8]

Consider a set of N independent experiments, each having two possible outcomes occurring with given probabilities.

$$\begin{array}{l|l} \text{events} & A + B = S \\ \text{probabilities} & p + q = 1 \\ \text{random variables} & n + m = N \end{array}$$

Binomial distribution:

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}.$$

Mean value: $\langle n \rangle = Np$.

Variance: $\langle \langle n^2 \rangle \rangle = Npq$. [nex15]

In the following we consider two different asymptotic distributions in the limit $N \rightarrow \infty$.

Poisson distribution:

Limit #1: $N \rightarrow \infty$, $p \rightarrow 0$ such that $Np = \langle n \rangle = a$ stays finite [nex15].

$$P(n) = \frac{a^n}{n!} e^{-a}.$$

Cumulants: $\langle \langle n^m \rangle \rangle = a$.

Factorial cumulants: $\langle \langle n^m \rangle \rangle_f = a \delta_{m,1}$. [nex16]

Single parameter: $\langle n \rangle = \langle \langle n^2 \rangle \rangle = a$.

Gaussian distribution:

Limit #2: $N \gg 1$, $p > 0$ with $Np \gg \sqrt{Npq}$.

$$P_N(n) = \frac{1}{\sqrt{2\pi \langle \langle n^2 \rangle \rangle}} \exp \left(-\frac{(n - \langle n \rangle)^2}{2 \langle \langle n^2 \rangle \rangle} \right).$$

Derivation: DeMoivre-Laplace limit theorem [nex21].

Two parameters: $\langle n \rangle = Np$, $\langle \langle n^2 \rangle \rangle = Npq$.

Special case of central limit theorem [nln9].