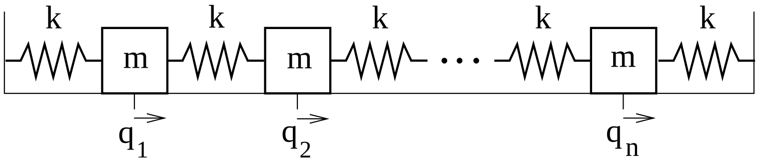
Elastic Chain [mln48]

Consider the elastic chain consisting of n blocks and $n+1$ springs as shown.



Equations of motion: $m\ddot{q}_j + k(2q_j - q_{j-1} - q_{j+1}) = 0, \quad j = 1, \ldots, n.$

Boundary conditions: $q_0(t) = q_{n+1}(t) = 0.$

Ansatz for solution: $q_j(t) = C\sin(\alpha j)\cos(\omega t + \phi), \quad j = 1, \ldots, n.$

Check boundary condition: $\sin[(n+1)\alpha] = 0 \quad \Rightarrow \alpha = \frac{\pi r}{n+1},\ r = 1, \ldots, n.$

Check eqs. of motion: $\left[-m\omega^2 + 2k\left(1 - \cos\frac{\pi r}{n+1}\right)\right] C\sin(\alpha j) = 0.$

Normal mode frequencies: $\omega_r = 2\sqrt{\frac{k}{m}}\sin\frac{\pi r}{2(n+1)}, \quad r = 1, \ldots, n.$

Normal mode amplitudes: $A_{jr} = \sin\left(\frac{\pi r}{n+1}\, j\right), \quad j, r = 1, \ldots, n.$

General solution: superposition of normal modes

$$\begin{aligned} q_j(t) &= \sum_{r=1}^{n} C_r \sin\left(\frac{\pi r}{n+1}\, j\right)\cos(\omega_r t + \phi_r) \\ &= \sum_{r=1}^{n} \sin\left(\frac{\pi r}{n+1}\, j\right)\left[d_r\cos\omega_r t + e_r\sin\omega_r t\right], \end{aligned}$$

with

$$\begin{aligned} d_r &= \frac{2}{n+1}\sum_{r=1}^{n} q_j(0)\sin\left(\frac{\pi r}{n+1}\, j\right), \\ e_r &= \frac{2}{n+1}\frac{1}{\omega_r}\sum_{r=1}^{n} \dot{q}_j(0)\sin\left(\frac{\pi r}{n+1}\, j\right). \end{aligned}$$