Solution by Quadrature [mln 4]

Any conservative, autonomous system with one degree of freedom is solvable by quadrature.

Equation of motion: $m\ddot{x} = F(x)$ (2^{nd} order ODE).

Solution by quadrature is a three-step process:

- Identify the first integral (conserved energy):
  Define $V(x) = -\int_{x_0}^{x} dx F(x), \quad F(x) = -\frac{dV}{dx}$;
  write $m\dot{x}\ddot{x} - \dot{x}F(x) = \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x)\right] = 0$;
  $\Rightarrow \; E = T + V = \frac{1}{2}m\dot{x}^2 + V(x) = \text{const.}$

- The first integral reduces the equation of motion to a 1^{st} order ODE:
  $\frac{dx}{dt} = \sqrt{2\left[E - V(x)\right]/m} \;\Rightarrow\; \int_0^t dt = \int_{x_0}^{x} \frac{dx}{\sqrt{2\left[E - V(x)\right]/m}}$.

- Invert the resulting function $t(x)$ to obtain the solution $x(t)$.

Application: harmonic oscillator

- $m\ddot{x} = -kx \;\Rightarrow\; m\dot{x}\ddot{x} + m\omega_0^2\dot{x}x = \frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2\right] = 0, \quad \omega_0^2 = \frac{k}{m}$.
  $\Rightarrow \; E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 = \text{const.}$

- $t = \int_{x_0}^{x} \frac{dx}{\sqrt{2E/m - \omega_0^2 x^2}} = \frac{1}{\omega_0}\arcsin\left(\frac{x\omega_0}{\sqrt{2E/m}}\right) + \text{const.}$

- $x(t) = \sqrt{\frac{2E}{m\omega_0^2}}\sin(\omega_0 t + \text{const})$.