

**[tex116] BE gas in  $\mathcal{D}$  dimensions V: heat capacity at low temperature**

The internal energy of the ideal Bose-Einstein gas in  $\mathcal{D} > 2$  dimensions and at  $T \leq T_c$  is given by the following expression:

$$\frac{U}{\mathcal{N}k_B T_v} = \frac{\mathcal{D}}{2} \zeta(\mathcal{D}/2 + 1) \left( \frac{T}{T_v} \right)^{\mathcal{D}/2+1}$$

(a) Use this result to derive the following expression for the heat capacity  $C_V = (\partial U / \partial T)_{V\mathcal{N}}$ :

$$\frac{C_V}{\mathcal{N}k_B} = \left( \frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4} \right) \frac{\zeta(\frac{\mathcal{D}}{2} + 1)}{\zeta(\frac{\mathcal{D}}{2})} \left( \frac{T}{T_c} \right)^{\mathcal{D}/2},$$

where  $T_c = T_v [\zeta(\mathcal{D}/2)]^{-2/\mathcal{D}}$  is the critical temperature and  $k_B T_v = \Lambda / v^{2/\mathcal{D}}$  with  $v \doteq V/\mathcal{N}$  and  $\Lambda \doteq h^2/2\pi m$  a convenient reference temperature. (b) Show that the heat capacity is continuous at  $T = T_c$  if  $\mathcal{D} \leq 4$  and discontinuous if  $\mathcal{D} > 4$ . Find the discontinuity  $\Delta C_V / \mathcal{N}k_B$  as a function of  $\mathcal{D}$  for  $\mathcal{D} > 4$ . (c) Infer from the result of [tex97] the leading singularity of  $C_V / \mathcal{N}k_B$  at  $T/T_v \ll 1$  for  $\mathcal{D} = 1$  and  $\mathcal{D} = 2$ . Then show that these singularities are consistent with the expression for  $C_V / \mathcal{N}k_B$  obtained here in part (a) provided we substitute  $(T_v/T_c)^{\mathcal{D}/2} = \zeta(\mathcal{D}/2)$ .

**Solution:**