# Ideal quantum gases:
# grand potential and thermal averages [tln64]

Grand potential: $\Omega(T,V,\mu) = -k_BT\ln Z = U - TS - \mu\mathcal{N} = -pV$.

- $\Omega_{MB} = -k_BT\sum_{k=1}^{\infty} ze^{-\beta\epsilon_k} = -k_BT\sum_{k=1}^{\infty} e^{-\beta(\epsilon_k-\mu)}$,
- $\Omega_{BE} = k_BT\sum_{k=1}^{\infty}\ln\left(1 - ze^{-\beta\epsilon_k}\right) = k_BT\sum_{k=1}^{\infty}\ln\left(1 - e^{-\beta(\epsilon_k-\mu)}\right)$,
- $\Omega_{FD} = -k_BT\sum_{k=1}^{\infty}\ln\left(1 + ze^{-\beta\epsilon_k}\right) = -k_BT\sum_{k=1}^{\infty}\ln\left(1 + e^{-\beta(\epsilon_k-\mu)}\right)$.

Parametric representation [$a=1$ (FD), $a=0$ (MB), $a=-1$ (BE)]:

$$\ln Z = \frac{pV}{k_BT} = \frac{1}{a}\sum_{k=1}^{\infty}\ln\left(1 + aze^{-\beta\epsilon_k}\right).$$

Average number of particles:

$$\mathcal{N} \overset{\text{[tln61]}}{=} \frac{1}{\beta}\frac{\partial\ln Z}{\partial\mu} = \sum_{k=1}^{\infty}\frac{1}{z^{-1}e^{\beta\epsilon_k}+a} = \sum_{k=1}^{\infty}\langle n_k\rangle.$$

Average energy (internal energy):

$$U = \sum_{k=1}^{\infty}\epsilon_k\langle n_k\rangle = \sum_{k=1}^{\infty}\frac{\epsilon_k}{z^{-1}e^{\beta\epsilon_k}+a}.$$

Average occupation number of energy level $\epsilon_k$:

$$\langle n_k\rangle = \frac{1}{e^{\beta(\epsilon_k-\mu)}+a} = -\beta^{-1}\frac{\partial\ln Z}{\partial\epsilon_k}.$$

Fluctuations in occupation number [tex110]:

$$\langle n_k^2\rangle - \langle n_k\rangle^2 = \beta^{-2}\frac{\partial^2\ln Z}{\partial\epsilon_k^2} = \frac{1}{Z}\beta^{-2}\frac{\partial^2 Z}{\partial\epsilon_k^2} - \left[\frac{1}{Z}\beta^{-1}\frac{\partial Z}{\partial\epsilon_k}\right]^2 = -\beta^{-1}\frac{\partial\langle n_k\rangle}{\partial\epsilon_k}.$$