Jacobi transformations [tln21]

Change of independent variables in a state function. Unlike in a Legendre transform, the state variable in question remains the same.

Task: Calculate $C_p = T\left(\frac{\partial S}{\partial T}\right)_p$ from entropy function $S(U,V)$.

Use $\left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T}$, $\left(\frac{\partial S}{\partial V}\right)_U = \frac{p}{T}$ and solve for $T(V,U), p(V,U)$.

Solving $T(V,U), p(V,U)$ for $V(T,p), U(T,p)$ to obtain $S(T,p)$ amounts to the simultaneous solution of two *nonlinear* equations for two variables.

Solving

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial V}\right)_U dV + \left(\frac{\partial T}{\partial U}\right)_V dU \\ dp &= \left(\frac{\partial p}{\partial V}\right)_U dV + \left(\frac{\partial p}{\partial U}\right)_V dU \end{aligned}$$

for

$$\begin{aligned} dV &= \frac{1}{J(V,U)}\left[\left(\frac{\partial p}{\partial U}\right)_V dT - \left(\frac{\partial T}{\partial U}\right)_V dp\right] \equiv \frac{J_1(V,U)}{J(V,U)} \\ dU &= \frac{1}{J(V,U)}\left[-\left(\frac{\partial p}{\partial V}\right)_U dT + \left(\frac{\partial T}{\partial V}\right)_U dp\right] \equiv \frac{J_2(V,U)}{J(V,U)} \end{aligned}$$

with Jacobian determinants

$$J(V,U) = \begin{vmatrix} \left(\frac{\partial T}{\partial V}\right)_U & \left(\frac{\partial T}{\partial U}\right)_V \\ \left(\frac{\partial p}{\partial V}\right)_U & \left(\frac{\partial p}{\partial U}\right)_V \end{vmatrix},$$

$$J_1(V,U) = \begin{vmatrix} dT & \left(\frac{\partial T}{\partial U}\right)_V \\ dp & \left(\frac{\partial p}{\partial U}\right)_V \end{vmatrix}, \quad J_2(V,U) = \begin{vmatrix} \left(\frac{\partial T}{\partial V}\right)_U & dT \\ \left(\frac{\partial p}{\partial V}\right)_U & dp \end{vmatrix},$$

amounts to the solution of two *linear* equations for two variables.

$$\begin{aligned} \Rightarrow\ dS &= \frac{1}{T}dU + \frac{p}{T}dV \\ &= \frac{1}{TJ}\left[-\left(\frac{\partial p}{\partial V}\right)_U + p\left(\frac{\partial p}{\partial U}\right)_V\right] dT + \frac{1}{TJ}\left[-p\left(\frac{\partial T}{\partial U}\right)_V + \left(\frac{\partial T}{\partial V}\right)_U\right] dp \end{aligned}$$

$$\Rightarrow\ C_p = T\left(\frac{\partial S}{\partial T}\right)_p = \frac{1}{J}\left[p\left(\frac{\partial p}{\partial U}\right)_V - \left(\frac{\partial p}{\partial V}\right)_U\right]$$