## [mex99] Hamilton's characteristic function for central force problem

Consider the one-body central-force problem specified by the Hamiltonian

$$H(r,p,\ell) = \frac{1}{2m}\left(p^2 + \frac{\ell^2}{r^2}\right) + V(r),$$

where $p \equiv p_r$ and $\ell \equiv p_\vartheta$ are the canonical momenta conjugate to $r$ and $\vartheta$, respectively. Solve the Hamilton-Jacobi equation for Hamilton's characteristic function. (a) Use the ansatz $W(r,\vartheta,\ell,E) = W_1(r,\ell,E) + \ell\vartheta$ for the characteristic function and determine $W_1(r,\ell,E)$. (b) The characteristic function $W(r,\vartheta,\ell,E)$ is the generating function of a canonical transformation $(r,\vartheta) \to (R,\Theta)$, which transforms the Hamiltonian as follows: $H(r,p,\ell) = K(E,\ell) = E$. Solve the canonical equations for $R,\Theta$. (c) Infer from $\partial W/\partial E = R = \text{const}$ the time evolution of the radial motion $r(t,E,\ell,r_0)$. (c) Infer from $\partial W/\partial \ell = \Theta = \text{const}$ the orbital relation $\vartheta(r,E,\ell,r_0,\vartheta_0)$, which, in combination with $r(t,E,\ell,r_0)$, determines the time evolution of the angular motion.

**Solution:**