[mex81] When does the Hamiltonian represent the total energy?

Consider a dynamical system with $3N$ degrees of freedom subject to k holonomic constraints: $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_n, t), i = 1, \ldots, N, n = 3N - k$. The kinetic and potential energies are given by the expressions

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2, \quad V = V(\mathbf{r}_1, \ldots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \ldots, \dot{\mathbf{r}}_N, t).$$

Show that the Hamiltonian $H(q_1, \ldots, q_n, p_1, \ldots, p_n, t)$ derived from these specifications is equal to the total energy, $E = T + V$, only if (i) the potential energy does not depend on the velocities $\dot{\mathbf{r}}_i$ and (ii) if the holonomic constraints are not explicitly time-dependent .

Solution: