[mex19] Hopf bifurcation

A simple Hopf bifurcation generates a limit cycle from a point attractor upon variation of some parameter in the equations of motion of a dynamical system. Consider the dynamical system specified (in polar coordinates) by

$$\dot{r} = -\Gamma r - r^3, \qquad \dot{\theta} = \omega,$$

where Γ and ω are constants.
(a) Find the exact solution $r(t), \theta(t)$ for initial conditions $r(0) = r_0, \theta(0) = 0$.
(b) Identify the circular periodic trajectory for $\Gamma < 0$, which plays the role of a limit cycle, and determine its radius.
(c) Determine the nature of the fixed point at $r = 0$ for $\Gamma > 0$ and $\Gamma < 0$.
(d) Plot three trajectories in the (x, y)-plane to illustrate the emergence of a limit cycle from a stable fixed point. The first trajectory is for $\Gamma > 0$. It will spiral into the point attractor at the origin. The second and third attractor are for $\Gamma < 0$ with initial conditions inside and outside the limit cycle, respectively. Fine-tune the parameters and initial conditions to make the message of your graph compelling.
(e) Choose several values of t_{max} for fixed values of r_0, ω, Γ. Then plot $r(t_{max})$ versus Γ to illustrate the emergence of a bifurcation singularity in the limit $t_{max} \to \infty$. Again fine-tune your parameter values to optimize your graph for didactic effect.

Solution: