[mex156] Spherical pendulum: Routhian and reduction to quadrature

A particle of mass m in a uniform gravitational field g is constrained to move on the surface of a sphere of radius ℓ.
(a) Find the Lagrangian $L(\theta, \phi, \dot{\theta}, \dot{\phi})$, where the range of the polar angle is $0 \leq \theta \leq \pi$ and the range of the azimuthal angle is $0 \leq \phi \leq 2\pi$.
(b) Derive the two Lagrange equations.
(c) Identify and determine the conserved quantity β_ϕ associated with the cyclic coordinate ϕ.
(d) Construct the Routhian in the form $R(\theta, \dot{\theta}, \beta_\phi) = T(\theta, \dot{\theta}, \beta_\phi) - V(\theta)$ and derive the Lagrange equation for the remaining dynamical variable θ from it.
(e) Use conservation of energy, $E = T(\theta, \dot{\theta}, \beta_\phi) + V(\theta)$ =const, to reduce the analytic solution of the spherical to quadrature following the steps in [mln4]. This includes a prescription of how to calculate the cyclic variable $\phi(t)$ from the integral that implicitly yields $\theta(t)$.

Solution: