[mex146] Plane pendulum III: librations

The plane pendulum consists of a point mass m constrained by a massless rod to move in a vertical circle of radius ℓ in a uniform gravitational field g.

(a) By reduction to quadrature find the angle $\theta(t)$ of the librational motion (at energy $E < 2mg\ell$). Establish the familiar result of harmonic oscillation for very low energies ($E \ll 2mg\ell$).

(b) Find the period of oscillation T as a function of energy. Expand the exact expression $T(E)$ at low energies and derive an expression $T(\theta_0)$, where θ_0 is the amplitude, which includes the leading anharmonic correction.

Solution: