[mex128] Apsidal angle reinterpreted

Consider a particle of mass m in a bounded orbit with energy E and angular momentum ℓ of a central force potential $V(r)$. Show that the angle $\Delta\vartheta$ between successive apsidal vectors (between pericenter and apocenter) is related to the period T of the oscillatory motion of a fictitious particle in a 1D potential $W(x)$ as investigated in [mex5]:

$$\Delta\vartheta \equiv \int_{r_{min}}^{r_{max}} dr\, \frac{\ell/mr^2}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{\ell^2}{2mr^2}\right]}} = \frac{T}{2\sqrt{m}}, \quad T = 2\int_{x_{min}}^{x_{max}} \frac{dx}{\sqrt{\frac{2}{m}\left[E - W(x)\right]}}.$$

Find the relation between the variables r and x and determine the function $W(x)$.

Solution: