Hamilton's Principal Function

We seek a canonical transformation $H(q, p, t) \rightarrow K(Q, P) \equiv 0$.
Here q stands for $q_1, \ldots, q_n$ etc.

Canonical equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \rightarrow \qquad \dot{Q}_i = \frac{\partial K}{\partial P_i} = 0, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0.$$

Hamilton's principal function: $S(q, P, t)$.

- S is the F_2-type generating function of this canonical transformation.
- S depends on $n + 1$ variables $q_1, \ldots, q_n, t$ and n parameters $P_1, \ldots, P_n$.
- $p_j = \dfrac{\partial S}{\partial q_j}, \quad Q_j = \dfrac{\partial S}{\partial P_j}, \quad K - H = \dfrac{\partial S}{\partial t}.$

Hamilton-Jacobi equation: $H\left(q_1, \ldots, q_n; \dfrac{\partial S}{\partial q_n}, \ldots, \dfrac{\partial S}{\partial q_n}; t\right) + \dfrac{\partial S}{\partial t} = 0.$

- First-order partial differential equation for $S(q, P, t)$.
- Integration constants $P_1, \ldots, P_n$ plus additive constant.
- For given solution: $p_j(q, P, t) = \dfrac{\partial S}{\partial q_j}, \quad Q_j(p, P, t) = \dfrac{\partial S}{\partial P_j} = \text{const.}$
- $\Rightarrow q_j(Q, P, t), \quad p_j(Q, P, t)$ (transformation relations).
- The constants Q_j, P_j are functions of the initial values $q_j^{(0)}, p_j^{(0)}$.

Physical significance of Hamilton's principal function:

$$\frac{dS}{dt} = \sum_j \frac{\partial S}{\partial q_j} \dot{q}_j + \frac{\partial S}{\partial t} = \sum_j p_j \dot{q}_j - H = L.$$