Canonical Transformations [mln89]

Canonical transformations $(q;p) \to (Q;P)$ operate in phase space.

Notation: $(q;p) \equiv (q_1,\ldots,q_n;p_1,\ldots,p_n)$ etc.

Not every transformation $q_i = q_i(Q;P;t)$, $p_i = p_i(Q;P;t)$ preserves the structure of the canonical equations.

Canonicity of transformation $(q;p) \to (Q;P)$ hinges on relation between Hamiltonians $H(q;p;t)$ and $K(Q;P;t)$ such that

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \Rightarrow \quad \dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}.$$

Canonicity enforced via modified Hamilton's principle [mln83]:

$$\delta\int_{t_1}^{t_2} dt\left[\sum_j p_j\dot{q}_j - H(q;p;t)\right] = \delta\int_{t_1}^{t_2} dt\left[\sum_j P_j\dot{Q}_j - K(Q;P;t)\right] = 0.$$

$$\Rightarrow \sum_j p_j\dot{q}_j - H(q;p;t) = \sum_j P_j\dot{Q}_j - K(Q;P;t) + \frac{dF}{dt}.$$

Generating function $F(x;Y;t)$ depends on n old and n new coordinates.
For example: $(x;Y) \equiv (q;Q)$, $(q;P)$, $(p;Q)$, $(p;P)$.

Total time derivative of F has vanishing variation:

$$\delta\int_{t_1}^{t_2} dt\,\frac{dF}{dt} = \Big[\delta F\Big]_{t_1}^{t_2} = 0.$$

Different generating functions for the same canonical transformation are related to each other via Legendre transform.

The four basic types of generating functions are

$$\begin{aligned} F_1(q;Q;t) &= F_2(q;P;t) - \sum_j P_jQ_j \\ &= F_3(p;Q;t) + \sum_j p_jq_j \\ &= F_4(p;P;t) - \sum_j P_jQ_j + \sum_j p_jq_j. \end{aligned}$$

Implementation of canonical transformation specified by $F_1(q;Q;t)$:

$$\frac{d}{dt}\,F_1(q;Q;t) = \sum_j (p_j\dot{q}_j - P_j\dot{Q}_j) - \left[H(q;p;t) - K(Q;P;t)\right].$$

$$\Rightarrow \quad \sum_j \left(\frac{\partial F_1}{\partial q_j}\,dq_j + \frac{\partial F_1}{\partial Q_j}\,dQ_j\right) + \frac{\partial F_1}{\partial t}\,dt \;=\; \sum_j (p_j dq_j - P_j dQ_j) \;-\; \left[H(q;p;t) - K(Q;P;t)\right]dt.$$

Comparison of coefficients yields

$$p_j = \frac{\partial F_1}{\partial q_j}, \quad P_j = -\frac{\partial F_1}{\partial Q_j}, \quad K - H = \frac{\partial F_1}{\partial t}.$$

Transformation relations:

- Invert relations $P_j(q;Q;t)$ into $q_j(Q;P;t)$.
- Combine relations $p_j(q;Q,t)$ with $q_j(Q;P;t)$ to get $p_j(Q;P;t)$.

Transformed Hamiltonian:

- $K(Q;P;t) = H(q;p;t) + \frac{\partial}{\partial t}\,F_1(q;Q;t)$.

****

generating function	**transformation of coordinates**		**transformation of Hamiltonian**
$F_1(q,Q,t)$	$p_j = \frac{\partial F_1}{\partial q_j}$	$P_j = -\frac{\partial F_1}{\partial Q_j}$	$K = H + \frac{\partial F_1}{\partial t}$
$F_2(q,P,t)$	$p_j = \frac{\partial F_2}{\partial q_j}$	$Q_j = \frac{\partial F_2}{\partial P_j}$	$K = H + \frac{\partial F_2}{\partial t}$
$F_3(p,Q,t)$	$q_j = -\frac{\partial F_3}{\partial p_j}$	$P_j = -\frac{\partial F_3}{\partial Q_j}$	$K = H + \frac{\partial F_3}{\partial t}$
$F_4(p,P,t)$	$q_j = -\frac{\partial F_4}{\partial p_j}$	$Q_j = \frac{\partial F_4}{\partial P_j}$	$K = H + \frac{\partial F_4}{\partial t}$