Hamiltonian and Canonical Equations [mln82]

Hamiltonian from Lagrangian via Legendre transform:

- Given the Lagrangian of a mechanical system: $L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t)$.
- Introduce canonical coordinates: $q_i,\ p_i \doteq \dfrac{\partial L}{\partial \dot{q}_i},\ i = 1, \ldots, n.$
- Construct Hamiltonian:

$$H(q_1, \ldots, q_n; p_1, \ldots, p_n; t) = \sum_j \dot{q}_j p_j - L(q_1, \ldots, q_n; \dot{q}_1, \ldots, \dot{q}_n; t),$$

  where $\dot{q}_j = \dot{q}_j(q_1, \ldots, q_n; p_1, \ldots, p_n; t)$ is inferred from $p_i = \partial L/\partial \dot{q}_i$.

Canonical equations from total differential of H:

- $dH = \sum_j \left[\dfrac{\partial H}{\partial q_j} dq_j + \dfrac{\partial H}{\partial p_j} dp_j \right] + \dfrac{\partial H}{\partial t} dt.$
- $d\left(\sum_j \dot{q}_j p_j - L \right) = \sum_j \left[\dot{q}_j dp_j + p_j d\dot{q}_j - \dfrac{\partial L}{\partial q_j} dq_j - \dfrac{\partial L}{\partial \dot{q}_j} d\dot{q}_j \right] - \dfrac{\partial L}{\partial t} dt;$

  use $\dfrac{\partial L}{\partial q_j} = \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = \dot{p}_j, \quad \dfrac{\partial L}{\partial \dot{q}_j} = p_j;$

  $\Rightarrow\ d\left(\sum_j \dot{q}_j p_j - L \right) = \sum_j \left[\dot{q}_j dp_j - \dot{p}_j dq_j \right] - \dfrac{\partial L}{\partial t} dt;$
- comparison of coefficients yields
  - $\dot{q}_j = \dfrac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\dfrac{\partial H}{\partial q_j}, \quad j = 1, \ldots, n$ (canonical equations),
  - $\dfrac{\partial H}{\partial t} = -\dfrac{\partial L}{\partial t}.$

Comments:

- The inversion of $p_i = \partial L/\partial \dot{q}_i$ as used above requires that $\det\left(\dfrac{\partial^2 L}{\partial \dot{q}_i \dot{q}_j} \right) \neq 0$ [mex189].
- Lagrangian from Hamiltonian: [mex188].