Lagrange equations derived from D'Alembert's principle [mln8]

D'Alembert's equation: $\sum_{j=1}^{3N-k}\left[\sum_{i=1}^{N}(m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i)\cdot\frac{\partial \mathbf{r}_i}{\partial q_j}\right]\delta q_j = 0$

- $\mathbf{F}_i(\mathbf{r}_j, \dot{\mathbf{r}}_j, t) \quad \Rightarrow \quad \sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} = Q_j, \quad j = 1,\ldots,3N-k.$

  $Q_j(q_1,\ldots,q_{3N-k},\dot{q}_1,\ldots,\dot{q}_{3N-k},t) \doteq \sum_{i=1}^{N}\mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j}.$

- $\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j}\right) - m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}\right].$

  $\frac{d\mathbf{r}_i}{dt} = \sum_{l=1}^{3N-k}\frac{\partial \mathbf{r}_i}{\partial q_l}\dot{q}_l + \frac{\partial \mathbf{r}_i}{\partial t} = \dot{\mathbf{r}}_i(q_j,\dot{q}_j,t) \;\Rightarrow\; \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j}.$

  $\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right).$

Kinetic energy: $T(q_j,\dot{q}_j,t) = \sum_{i=1}^{N}\frac{1}{2}m_i\dot{\mathbf{r}}_i^2.$

$\Rightarrow \sum_{j=1}^{3N-k}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j\right)\delta q_j = 0$ with independent δq_j.

$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad j = 1,\ldots,3N-k.$

Assumption: $\mathbf{F}_i = -\nabla_i\tilde{V}, \quad \tilde{V}(\mathbf{r}_1,\ldots,\mathbf{r}_N,t) = V(q_1,\ldots,q_{3N-k},t).$

$\Rightarrow Q_j = Q_j(q_1,\ldots,q_{3N-k},t) = -\frac{\partial V}{\partial q_j}, \quad \frac{\partial V}{\partial \dot{q}_j} = 0.$

Lagrangian: $L(q_j,\dot{q}_j,t) \doteq T(q_j,\dot{q}_j,t) - V(q_j,t).$

Lagrange equations: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1,\ldots,3N-k.$

Generalized momenta: $p_j \doteq \frac{\partial L}{\partial \dot{q}_j}, \quad j = 1,\ldots,3N-k.$