Laplace-Runge-Lenz Vector [mln45]

Central force: $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$.

Equation of motion: $\dot{\mathbf{p}} = F(r)\dfrac{\mathbf{r}}{r}$.

Angular momentum: $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}$.

Conservation law: $\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = 0 \;\Rightarrow\; \mathbf{L} = \text{const.}$

$$\begin{aligned} \Rightarrow \frac{d}{dt}(\mathbf{p} \times \mathbf{L}) &= \dot{\mathbf{p}} \times \mathbf{L} = \frac{mF(r)}{r}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{mF(r)}{r}\left[\mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - r^2\dot{\mathbf{r}}\right] \\ &= -mF(r)r^2\left[\frac{1}{r}\dot{\mathbf{r}} - \frac{\dot{r}}{r^2}\mathbf{r}\right] = -mF(r)r^2\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right]. \end{aligned}$$

We have used $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$, $\quad \mathbf{r} \cdot \dot{\mathbf{r}} = \dfrac{1}{2}\dfrac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = r\dot{r}$.

Kepler system: $F(r) = -\dfrac{\kappa}{r^2} \;\Rightarrow\; \dfrac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \dfrac{d}{dt}\left[\dfrac{m\kappa\mathbf{r}}{r}\right]$.

Laplace-Runge-Lenz vector: $\mathbf{A} = \mathbf{p} \times \mathbf{L} - m\kappa\dfrac{\mathbf{r}}{r} = \text{const.}$

The vector $\mathbf{A}$ lies in the plane of the orbit, points to the pericenter, and has magnitude $A = m\kappa e$, where e is the eccentricity of the orbit.

$$\mathbf{L} \perp \mathbf{p} \times \mathbf{L} \text{ and } \mathbf{L} \perp \mathbf{r} \;\Rightarrow\; \mathbf{A} \perp \mathbf{L}.$$

$$\mathbf{A} \cdot \mathbf{r} \equiv Ar\cos\vartheta = \mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) - m\kappa r = \mathbf{L} \cdot (\mathbf{r} \times \mathbf{p}) - m\kappa r = \ell^2 - m\kappa r.$$

$$\mathbf{A} \cdot \frac{\mathbf{r}}{r} \text{ assumes its maximum value for } r = r_{min} \quad \text{(pericenter).}$$

$$\frac{\ell^2/m\kappa}{r} = 1 + \frac{A}{m\kappa}\cos\vartheta \quad \text{(conic section).}$$