Routhian Function [mln39]

Goal: systematic elimination of cyclic coordinates in the Lagrangian formulation of mechanics.

Consider a system with n generalized coordinates of which the first k are cyclic.

Lagrangian: $L(q_{k+1},\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t) \;\Rightarrow\; q_1,\ldots,q_k$ are cyclic.

Routhian: $R(q_{k+1},\ldots,q_n,\dot{q}_{k+1},\ldots,\dot{q}_n,\beta_1,\ldots,\beta_k,t) = L - \sum_{i=1}^{k}\beta_i\dot{q}_i.$

where the relations $\beta_i = \dfrac{\partial L}{\partial \dot{q}_i} = \text{const},\; i=1,\ldots,k$ are to be inverted into
$\dot{q}_i = \dot{q}_i(q_{k+1},\ldots,q_n,\dot{q}_{k+1},\ldots,\dot{q}_n,\beta_1,\ldots,\beta_k,t), \quad i=1,\ldots,k.$

Compare coefficients of the variations

$$\delta R = \sum_{i=k+1}^{n}\frac{\partial R}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n}\frac{\partial R}{\partial \dot{q}_i}\delta\dot{q}_i + \sum_{i=1}^{k}\frac{\partial R}{\partial \beta_i}\delta\beta_i + \frac{\partial R}{\partial t}\delta t,$$

$$\delta\left(L - \sum_{i=1}^{k}\beta_i\dot{q}_i\right) = \sum_{i=k+1}^{n}\frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=k+1}^{n}\frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i - \sum_{i=1}^{k}\dot{q}_i\delta\beta_i + \frac{\partial L}{\partial t}\delta t.$$

Resulting relations between partial derivatives:

$$\frac{\partial R}{\partial q_i} = \frac{\partial L}{\partial q_i}, \quad \frac{\partial R}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}, \quad i=k+1,\ldots,n,$$

$$\frac{\partial R}{\partial t} = \frac{\partial L}{\partial t}; \quad \dot{q}_i = -\frac{\partial R}{\partial \beta_i}, \quad i=1,\ldots,k.$$

Lagrange equations for the noncyclic coordinates:

$$\frac{\partial R}{\partial q_i} - \frac{d}{dt}\frac{\partial R}{\partial \dot{q}_i} = 0, \quad i=k+1,\ldots,n.$$

Time evolution of cyclic coordinates:

$$q_i(t) = -\int dt\,\frac{\partial R}{\partial \beta_i}, \quad i=1,\ldots,k.$$