Holonomic Constraints [mln36]

Consider a system of N particles moving in 3D space.
$3N$ Cartesian coordinates: $\mathbf{r}_i = (x_i, y_i, z_i),\ i = 1, \ldots, N$.

Equations of motion: $m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(ext)} + \sum_{j \neq i} \mathbf{F}_{ji},\ i = 1, \ldots, N$.

Some of the forces may be due to constraints and are not given.

Consider k holonomic constraints: $f_j(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = 0,\ j = 1, \ldots, k$.
The system is then said to have $3N-k$ degrees of freedom. Its configuration space is a $(3N-k)$-dimensional manifold.

Holonomic constraints that do not depend on time are named *scleronomic* (rigid), those that do depend on time are named *rheonomic* (flowing).

Transformation to *generalized coordinates*: $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_n, t),\ n = 3N-k$.

- Plane pendulum: $N = 1,\ k = 2 \ \Rightarrow\ n = 1$.
  Constraints: $z = 0,\ x^2 + y^2 = L^2$.
  Generalized coordinate ϕ: $\mathbf{r} = (L\sin\phi, L\cos\phi, 0)$.
- Rigid body (made of N atoms):
  Constraints: $(\mathbf{r}_i - \mathbf{r}_j)^2 = c_{ij} = \text{const},\ i, j = 1, \ldots, N$.
  Degrees of freedom: $n = 6$ (3 translations plus 3 rotations).

Q: What is the general structure of the equations of motion for the generalized coordinates?
A: A set of n 2^{nd} order ODEs, named *Lagrange equations*, for the n generalized coordinates q_i.

Q: Do the Lagrange equations depend on the forces of constraint?
A: Not explicitly. They can be solved without knowledge of the forces of constraint.

Q: What if I wish to know the forces of constraint?
A: They can be determined either from the solution of the Lagrange equations or from a modified set of $n+k$ equations of motion that yield the time evolution of the generalized coordinates and the forces of constraint simultaneously.

Q: How are the Lagrange equations derived?
A: Lagrange's equations can be derived from Newton's equations by invoking *D'Alembert's principle*, which exploits the fact that the forces of constraint do not perform any work. Lagrange's equations can also be inferred from *Hamilton's principle*, an extremum principle whose scope is wider than that of Newtonian mechanics.