Angular Momentum [mln26]

$$\mathbf{L}_{tot} = \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}_\alpha) \times \left(\dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_\alpha\right).$$

If the rigid body rotates freely, choose the origin of the body frame at the center of mass: $\sum_\alpha m_\alpha \mathbf{r}_\alpha = 0,\ \mathbf{R} = \mathbf{r}_{cm},\ \dot{\mathbf{R}} = \mathbf{v}_{cm}$.

$$\Rightarrow\ \mathbf{L}_{tot} = m\mathbf{r}_{cm} \times \mathbf{v}_{cm} + \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha) = \mathbf{L}_{orb} + \mathbf{L}_{spin}.$$

If the rigid body rotates about a fixed point in the inertial frame, choose the origin of the body and inertial frames at the fixed point: $\mathbf{R} = 0,\ \dot{\mathbf{R}} = 0$.

$$\Rightarrow\ \mathbf{L}_{tot} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha) = \mathbf{L}_{spin}.$$

Consider a rigid body undergoing a purely rotational motion ($\dot{\mathbf{R}} = 0$).

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha) = \sum_\alpha m_\alpha \left[r_\alpha^2\, \vec{\omega} - \mathbf{r}_\alpha (\mathbf{r}_\alpha \cdot \vec{\omega})\right].$$

Use body frame components $\omega_1, \omega_2, \omega_3$ and $r_{\alpha 1}, r_{\alpha 2}, r_{\alpha 3}$:

$$L_i = \sum_\alpha m_\alpha \left[r_\alpha^2 \omega_i - r_{\alpha i} \sum_j r_{\alpha j} \omega_j\right] = \sum_\alpha m_\alpha \sum_j \left[\omega_j \delta_{ij} r_\alpha^2 - \omega_j r_{\alpha i} r_{\alpha j}\right].$$

$$\Rightarrow\ L_i = \sum_j \omega_j \sum_\alpha m_\alpha \left[\delta_{ij} r_\alpha^2 - r_{\alpha i} r_{\alpha j}\right] = \sum_j I_{ij}\, \omega_j.$$

Comments:

- The vectors $\vec{\omega}$ and $\mathbf{L}$ are, in general, not parallel.
- Matrix notation: $\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$.
- Kinetic energy: $T = \frac{1}{2}\vec{\omega} \cdot \mathbf{L}$.
- If $\vec{\omega} = \text{const}$ then $\mathbf{L}$ varies $\Rightarrow$ $\mathbf{N} \neq 0$ (torque).
- If $\mathbf{N} = 0$ then $\mathbf{L} = \text{const}$ $\Rightarrow$ $\vec{\omega}$ varies.
- If the body frame is along principal axes, then $I_{ij} = I_i \delta_{ij}$.
  $\Rightarrow\ L_i = I_i \omega_i,\ T = \frac{1}{2}\sum_i I_i \omega_i^2$.