Motion in time on elliptic Kepler orbit [mln19]

Use the formal solution with $E < 0,\ V(r) = -\frac{\kappa}{r},\ \kappa = GmM$:

$$t = \int \frac{dr}{\sqrt{\frac{2}{m}\left[E + \frac{\kappa}{r} - \frac{\ell^2}{2mr^2}\right]}} = \sqrt{\frac{m}{2|E|}} \int \frac{rdr}{\sqrt{-r^2 + \frac{\kappa}{|E|}\, r - \frac{\ell^2}{2m|E|}}}.$$

Introduce semi-major axis $a = \dfrac{\kappa}{2|E|}$ and eccentricity $e = \sqrt{1 - \dfrac{2|E|\ell^2}{m\kappa^2}}$:

$$\Rightarrow\ t = \sqrt{\frac{ma}{\kappa}} \int \frac{rdr}{\sqrt{a^2e^2 - (a-r)^2}}.$$

Substitute $a - r = ae\cos\psi$: $\Rightarrow\ r(\psi) = a(1 - e\cos\psi), \quad dr = ae\sin\psi d\psi$.

$$\Rightarrow\ t = \sqrt{\frac{ma}{\kappa}} \int d\psi\, a(1 - e\cos\psi) = \sqrt{\frac{ma^3}{\kappa}}(\psi - e\sin\psi).$$

For the angular coordinate ϑ, use $r(\psi)$ and the orbital equation $r(\vartheta) = p/(1+\cos\vartheta)$ with $p = a(1-e^2)$. Then eliminate r from $r(\vartheta)$ and $r(\psi)$. For the Cartesian coordinates use $ex = p - r$ and $x^2 + y^2 = r^2$.

Parametric representation for the motion in time: $0 \leq \psi \leq 2\pi$

$$\begin{aligned}
r(\psi) &= a(1 - e\cos\psi) && (r_{min} \leq r \leq r_{max}) \\
\tan\frac{\vartheta(\psi)}{2} &= \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2} && (0 \leq \vartheta \leq 2\pi) \\
x(\psi) &= a(\cos\psi - e) \\
y(\psi) &= a\sqrt{1-e^2}\sin\psi \\
t(\psi) &= \sqrt{\frac{ma^3}{\kappa}}(\psi - e\sin\psi) && (0 \leq t \leq \tau)
\end{aligned}$$

Period of motion: $\tau = 2\pi\sqrt{\dfrac{ma^3}{\kappa}}$.

Circular limit: $e = 0\ \Rightarrow\ r = a = \text{const},\ \vartheta = \psi,\ t = \tau\,\dfrac{\vartheta}{2\pi}$.