Generalized Forces of Constraint in Lagrangian Mechanics [mln15]

Lagrangian: $L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$.

Differential constraints: $\displaystyle\sum_{i=1}^{n} a_{ji} dq_i + a_{jt} dt = 0, \quad j = 1, \ldots, m.$

Relations between virtual displacements: $\displaystyle\sum_{i=1}^{n} a_{ji} \delta q_i = 0, \quad j = 1, \ldots, m.$

The generalized forces of constraint, Q_i, do not perform any work.

D'Alembert's principle $\Rightarrow$ $\displaystyle\sum_{i=1}^{n} Q_i \delta q_i = 0.$

$\Rightarrow \displaystyle\sum_{i=1}^{n} \left(Q_i - \sum_{j=1}^{m} \lambda_j a_{ji} \right) \delta q_i = 0$ for arbitrary values of λ_j.

Choose the Lagrange multipliers λ_j to satisfy $Q_i = \displaystyle\sum_{j=1}^{m} \lambda_j a_{ji}, \; i = 1, \ldots, n.$

The δq_i can now be chosen independently because the constraints are enforced by the generalized forces Q_i.

The solution of the dynamical problem is then determined by the following $n + m$ equations for the n dynamical variables q_i and the m Lagrange multipliers λ_j:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \sum_{j=1}^{m} \lambda_j a_{ji} = 0, \quad i = 1, \ldots, n.$$

$$\sum_{i=1}^{n} a_{ji} \dot{q}_i + a_{jt} = 0, \quad j = 1, \ldots, m.$$

For holonomic constraints, $f_j(q_1, \ldots, q_n, t) = 0, \; j = 1, \ldots, m$, we have

$$a_{ji} = \frac{\partial f_j}{\partial q_i}, \quad a_{jt} = \frac{\partial f_j}{\partial t}, \quad Q_i = \sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial q_i}.$$

Whereas holonomic constraints can be handled *kinematically*, i.e. via the elimination of redundant coordinates, nonholonomic constraints must be handled *dynamically*, i.e. via the explicit use of constraint forces.

In some cases, the generalized forces of constraint Q_j can be determined without integrating the equations of motion.