Constants of the Motion [mln10]

Consider a dynamical system with n degrees of freedom and described by generalized coordinates $q_1, \ldots, q_n$.

Definition: Any function $f(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$ with $df/dt = 0$ is a *constant of the motion*.

The terms *constant of the motion, conserved quantity, invariant, integral of the motion* are used interchangeably in the literature.

Fact: Any system with n degrees of freedom has $2n$ constants of the motion.

Obvious choices are the $2n$ initial conditions of the Lagrange equations (2^{nd} order ODEs): $c_k(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t),\ k = 1, \ldots, 2n$ derived from $q_i(c_1, \ldots, c_{2n}, t), \dot{q}_i(c_1, \ldots, c_{2n}, t),\ i = 1, \ldots, n$.

Comments:

- Not all constants of the motion are equally important. Some have a stronger impact on the time evolution of the system than others.
- Constants of the motion which are not known prior to the analytic solution of the system are, in general, not useful.
- Integration constants in particular (initial conditions, boundary values) are only meaningful constants of the motion if an analytic solution is available.
- Meaningful constants of the motion may very well be identified in systems for which no analytic solution exists.
- Certain constants of the motion can be used to reduce the number of degrees of freedom by factoring out single degrees of freedom (one per invariant). These special constants of the motion are best described in the context of Hamiltonian mechanics.
- The existence of n such constants of the motion preclude the dynamical system from behaving chaotically. Such system are called integrable.
- Some constants of the motion can be derived from known symmetries of the dynamical system, others point to obscure or hidden symmetries.