[tex93] Relativistic ideal gas (heat capacity)

(a) Derive from the result for the internal energy $U(T,N)$ of the relativistic classical ideal gas as calculated in [tex92] the heat capacity in the form

$$C_V(T,N) = Nk_B u\left[u + \frac{3}{u} - \frac{K_1(u)}{K_2(u)}\left(3 + u\frac{K_1(u)}{K_2(u)}\right)\right], \quad u \equiv \beta mc^2, \quad \beta = \frac{1}{k_B T}.$$

by using the recursion relations $K_n'(u) = -K_{n-1}(u) - (n/u)K_n(u)$ and $K_{n-1}(u) = K_{n+1}(u) - (2/u)K_n(u)$ for modified Bessel functions.
(b) Use the asymptotic results

$$\frac{K_1(u)}{K_2(u)} \simeq 1 - \frac{3}{2u} + \frac{15}{8u^2} \quad (u \gg 1), \qquad \frac{K_1(u)}{K_2(u)} \simeq \frac{u}{2} \quad (u \ll 1)$$

to recover the the results $C_V = \frac{3}{2}Nk_B$ and $C_V = 3Nk_B$ in the nonrelativistic and ultrarelativistic limits, respectively.

Solution: