[tex203] Ising model in Bethe approximation

Start from the expression,

$$Z_c = e^{\beta H}\Big[2\cosh\big(\beta[J+H']\big)\Big]^z + e^{-\beta H}\Big[2\cosh\big(\beta[J-H']\big)\Big]^z, \tag{1}$$

for the canonical partition function of a cluster consisting of central site 0 and z surrounding nearest-neighbor sites. Here J is the strength of the neaest-neighbor Ising coupling, H is the magnetic field, H' is the effective field, z is the coordination number, and $\beta = 1/k_BT$. Carry out all tasks in a Mathematica notebook.

(a) Derive expressions for the average values of the central spin and one nearest-neighbor spin via

$$\langle\sigma_0\rangle = \frac{\partial}{\partial H}\Big(k_BT\ln Z_c\Big), \qquad \langle\sigma_j\rangle = \frac{1}{z}\frac{\partial}{\partial H'}\Big(k_BT\ln Z_c\Big) \tag{2}$$

and express them as functions of β, H', J for $H=0$.

(b) Show that the requirement of translational invariance, $\langle\sigma_0\rangle = \langle\sigma_j\rangle$, implies the following relation between the dimensionless variables βJ and H'/J:

$$e^{2\beta H'} = \Big[\cosh\big(\beta[J+H']\big)\Big]^{z-1}\Big[\cosh\big(\beta[J-H']\big)\Big]^{1-z}. \tag{3}$$

(c) The existence of an ordered state, $\langle\sigma_0\rangle \neq 0$, implies that (3) has a real solution for $H'\neq 0$. Plot the left-hand side and the right-hand side of (3) as functions of H' for several values of β to demonstrate that a real solution at $H'\neq 0$ exists if T is sufficiently low. That solution will gradually move toward $H'=0$. If it disappears before reaching that value, the transition is discontinuous (first order), otherwise it is continuous (second order).

(d) Show that if the transition is of second order as the evidence suggests, the transition temperature is the solution of

$$(z-1)\tanh(\beta_cJ) = 1. \tag{4}$$

Plot k_BT_c/J versus z for $0<z<7$ and compare the result with the mean-field transition temperature $k_BT_{MF}/J = z$.

(f) Plot the order parameter $\langle\sigma_0\rangle$ from (2) as a function of k_BT/J with $H'(\beta)$ from (3) imposed. Show plots for various values of z including $z=4$ (square lattice) and $z=6$ (simple cubic lattice).

Solution: