

[tex189] Ising chain: transfer matrix solution II

Consider an Ising chain of N sites with periodic boundary conditions:

$$\mathcal{H} = - \sum_{l=1}^N \left[J \sigma_l \sigma_{l+1} + \frac{1}{2} H (\sigma_l + \sigma_{l+1}) \right], \quad \sigma_l = \pm 1, \quad \sigma_{N+1} = \sigma_1.$$

The canonical partition function Z_N as inferred from the trace of the transfer matrix is calculated in [tex185] and various thermodynamic quantities inferred from it. The task here is to evaluate the following expectation values along the lines outlined in [tsc18]:

(a) the expectation value, $\langle \sigma_i \rangle = \frac{1}{Z_N} \text{Tr}[\mathbf{S} \cdot \mathbf{V}^N]$, $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{V} = \begin{pmatrix} e^{\hat{J} + \hat{H}} & e^{-\hat{J}} \\ e^{-\hat{J}} & e^{\hat{J} - \hat{H}} \end{pmatrix}$;

(b) the correlation function, $\langle \sigma_i \sigma_{i+n} \rangle = \frac{1}{Z_N} \text{Tr}[\mathbf{S} \cdot \mathbf{V}^n \cdot \mathbf{S} \cdot \mathbf{V}^{N-n}]$;

(c) the covariance, $C(n) \doteq \langle \sigma_i \sigma_{i+n} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+n} \rangle$.

Solution: