[tex185] Ising chain: transfer matrix solution I

Consider an Ising chain of N sites with periodic boundary conditions:

$$\mathcal{H} = -\sum_{l=1}^{N}\Big[J\sigma_l\sigma_{l+1} + \frac{1}{2}H(\sigma_l + \sigma_{l+1})\Big], \qquad \sigma_l = \pm 1, \quad \sigma_{N+1} = \sigma_1. \tag{1}$$

The canonical partition function Z_N is inferred from the trace of the transfer matrix $\mathbf{V}$ [tsc18]:

$$Z_N = \mathrm{Tr}\big[\mathbf{V}^N\big], \qquad \mathbf{V} = \left(\begin{array}{cc} e^{\hat{J}+\hat{H}} & e^{-\hat{J}} \\ e^{-\hat{J}} & e^{\hat{J}-\hat{H}} \end{array}\right), \quad \hat{J} \doteq \frac{J}{k_BT}, \quad \hat{H} \doteq \frac{H}{k_BT}. \tag{2}$$

(a) Calculate the two eigenvalues $\lambda_\pm$ of $\mathbf{V}$.
(b) Derive an analytic function for the Gibbs free energy per lattice site as follows:

$$\bar{G}(T,H) = -k_BT \lim_{N\to\infty} N^{-1}\ln Z_N = -k_BT\ln\lambda_+. \tag{3}$$

(c) Derive from (3) compact analytic expressions for the magnetization $\bar{M}(T,H)$, the entropy $\bar{S}(T,H)$, the internal energy $\bar{U}(T,H)$, the enthalpy $\bar{E}(T,H)$, the heat capacity $\bar{C}_H(T,H)$, and the susceptibility $\bar{\chi}_T(T,H)$ (all per lattice site) using definitions and general thermodynamic relations.

Solution: