[tex172] Ideal lattice gas

Consider an array of N_c cells of equal shape forming a lattice fixed in space. Each cell has volume v_c and can be either vacant or occupied by exactly one particle with energy $-\mu$.
(a) Reinterpret the canonical partition function for this systems of cells,

$$Z_{N_c} = (1 + e^{\beta\mu})^{N_c}, \quad \beta \doteq \frac{1}{k_B T}, \tag{1}$$

as the grand partition function for an open system of noninteracting particles (named *ideal lattice gas*), where the volume is $V = N_c v_c$, and where μ is the chemical potential. Derive the grand potential $\Omega(T, V, \mu)$ from (1) thus understood.
(b) Derive from $\Omega(T, V, \mu)$ expressions for the pressure p, the average number of particles N, and the entropy S as functions of T, V, μ.
(c) Extract from the expressions for p and N the equation of state in the form $pV/Nk_BT = f(N/N_c)$ and show that for $N \ll N_c$ the classical-ideal-gas equation of state, $pV = Nk_BT$, emerges.
(d) Describe in words the deviation of pV/Nk_BT from unity in the ideal-lattice-gas equation of state as N/N_c increases toward unity and supply a physical interpretation for this deviation.
(e) Show that from the expressions for S and N derived in part (b) the following functional dependence of S on N/N_c follows:

$$\frac{S}{N_c k_B} = -\frac{N}{N_c} \ln \frac{N}{N_c} - \left(1 - \frac{N}{N_c}\right) \ln \left(1 - \frac{N}{N_c}\right).$$

(f) Sketch a curve of $S/N_c k_B$ versus N/N_c and interpret the shape of this curve.

Solution: