[tex116] BE gas in $\mathcal{D}$ dimensions V: heat capacity at low temperature

The internal energy of the ideal Bose-Einstein gas in $\mathcal{D} > 2$ dimensions and at $T \leq T_c$ is given by the following expression:

$$\frac{U}{\mathcal{N} k_B T_v} = \frac{\mathcal{D}}{2}\zeta(\mathcal{D}/2+1)\left(\frac{T}{T_v}\right)^{\mathcal{D}/2+1} \quad : \ \mathcal{D} > 2.$$

(a) Use this result to derive the following expression for the heat capacity $C_V = (\partial U/\partial T)_{V\mathcal{N}}$:

$$\frac{C_V}{\mathcal{N} k_B} = \left(\frac{\mathcal{D}}{2} + \frac{\mathcal{D}^2}{4}\right)\frac{\zeta\left(\frac{\mathcal{D}}{2}+1\right)}{\zeta\left(\frac{\mathcal{D}}{2}\right)}\left(\frac{T}{T_c}\right)^{\mathcal{D}/2} \quad : \ \mathcal{D} > 2,$$

where $T_c = T_v[\zeta(\mathcal{D}/2)]^{-2/\mathcal{D}}$ is the critical temperature and $k_B T_v = \Lambda/v^{2/\mathcal{D}}$ with $v \doteq V/\mathcal{N}$ and $\Lambda \doteq h^2/2\pi m$ a convenient reference temperature.
(b) Show that the heat capacity is continuous at $T = T_c$ if $\mathcal{D} \leq 4$ and discontinuous if $\mathcal{D} > 4$. Find the discontinuity $\Delta C_V/\mathcal{N} k_B$ as a function of $\mathcal{D}$ for $\mathcal{D} > 4$.
(c) Infer from the result of [tex97] the leading singularity of $C_V/\mathcal{N} k_B$ at $T/T_v \ll 1$ for $\mathcal{D} = 1$ and $\mathcal{D} = 2$. Then show that these singularities are consistent with the expression for $C_V/\mathcal{N} k_B$ obtained here in part (a) provided we substitute $(T_v/T_c)^{\mathcal{D}/2} = \zeta(\mathcal{D}/2)$.

Solution: