[tex111] Density of energy levels for ideal quantum gas in $\mathcal{D}$ dimensions

(a) Consider a nonrelativistic ideal quantum gas in $\mathcal{D}$ dimensions and confined to a box of volume $V = L^{\mathcal{D}}$ with rigid walls. The energy-momentum relation is $\epsilon = \hbar^2|\mathbf{k}|^2/2m$ and the density of states in reciprocal space is $\rho(\mathbf{k}) = (L/2\pi)^{\mathcal{D}} = \text{const}$. Show that the density of energy levels is

$$D(\epsilon) = \frac{V}{\Gamma(\mathcal{D}/2)} \left(\frac{2\pi m}{h^2}\right)^{\mathcal{D}/2} \epsilon^{\mathcal{D}/2-1}.$$

(b) Consider an ultrarelativistic ideal quantum gas in $\mathcal{D}$ dimensions and confined to a box of volume $V = L^{\mathcal{D}}$ with rigid walls. The energy-momentum relation is $\epsilon = \hbar|\mathbf{k}|c$ and the density of states in reciprocal space is $\rho(\mathbf{k}) = (L/2\pi)^{\mathcal{D}} = \text{const}$. Show that the density of energy levels is

$$D(\epsilon) = \frac{V\mathcal{A}_{\mathcal{D}}}{(hc)^{\mathcal{D}}}\,\epsilon^{\mathcal{D}-1} = \frac{2V\pi^{\mathcal{D}/2}}{\Gamma(\mathcal{D}/2)(ch)^{\mathcal{D}}}\,\epsilon^{\mathcal{D}-1}, \quad \mathcal{A}_{\mathcal{D}} = \frac{2\pi^{\mathcal{D}/2}}{\Gamma(\mathcal{D}/2)},$$

where $\mathcal{A}_{\mathcal{D}}$ is the area of the unit sphere in $\mathcal{D}$ dimensions.
(c) Generalize these results to the case of the relativistic energy-momentum relation,

$$\epsilon = \sqrt{m^2c^4 + (\hbar kc)^2} - mc^2,$$

and show that the previous two results are recovered in the appropriate limits of the general expression for $D(\epsilon)$.

Solution: