

# Bose–Einstein functions [ts136]

Bose-Einstein functions are polylogarithm functions:

$$g_n(z) = \text{Li}_n(z) \doteq \frac{1}{\Gamma(n)} \int_0^\infty \frac{dx x^{n-1}}{z^{-1}e^x - 1} = \sum_{l=1}^{\infty} \frac{z^l}{l^n}, \quad 0 \leq z \leq 1.$$

Special cases:

$$g_0(z) = \frac{z}{1-z}, \quad g_1(z) = -\ln(1-z), \quad g_\infty(z) = z.$$

Riemann zeta function:

$$g_n(1) = \zeta(n) \doteq \sum_{l=1}^{\infty} \frac{1}{l^n}.$$

Special values:

$$\zeta(1) \rightarrow \infty, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.$$

Recurrence relation:

$$z g'_n(z) = g_{n-1}(z), \quad n \geq 1.$$

Singularity at  $z = 1$  for non-integer  $n$ :

$$g_n(\alpha) = \Gamma(1-n)\alpha^{n-1} + \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \zeta(n-\ell)\alpha^\ell, \quad \alpha \doteq -\ln z.$$

