Legendre transform [tln77]

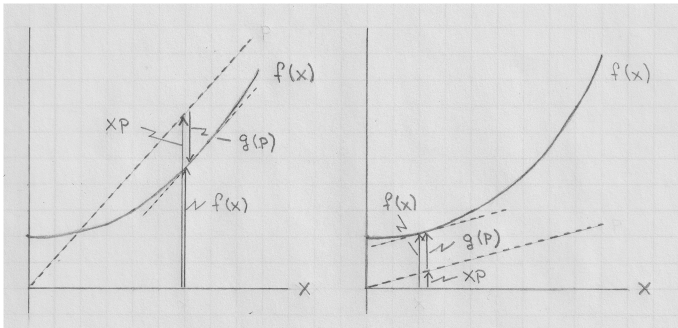
The theory of Legendre transforms is well developed for a wide range of applications. Here we limit the discussion to a key attribute.

Given is a function $f(x)$ with monotonic derivative $f'(x)$. The goal is to replace the independent variable x by $p = f'(x)$ with no loss of information.

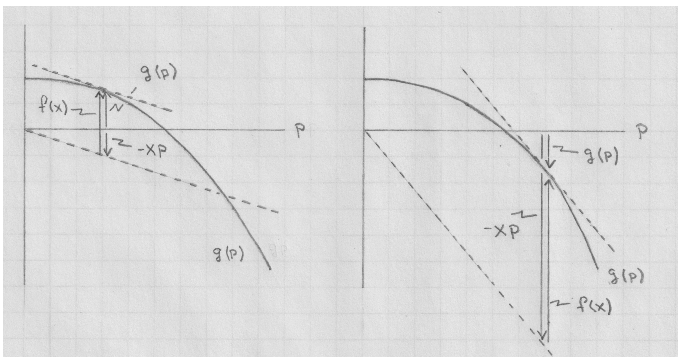
No loss of information means that the transformation must be invertible. The function $G(p) = f(x)$ with $p = f'(x)$ is, in general, not invertible.

The Legendre transform circumnavigates this problem elegantly.

▷ Forward direction: $g(p) = f(x) - xp$ with $p = f'(x)$.



◁ Reverse direction: $f(x) = g(p) + px$ with $x = -g'(p)$



Example 1: $f(x) = x^2 + 1$.

$$\triangleright\ f(x) = x^2 + 1 \quad \Rightarrow\ f'(x) = 2x \quad \Rightarrow\ x = \frac{p}{2} \quad \Rightarrow\ g(p) = 1 - \frac{p^2}{4}.$$

$$\triangleleft\ g(p) = 1 - \frac{p^2}{4} \quad \Rightarrow\ g'(p) = -\frac{p}{2} \quad \Rightarrow\ p = 2x \quad \Rightarrow\ f(x) = x^2 + 1.$$

Example 2: $f(x) = e^{2x}$.

$$\triangleright \; f(x) = e^{2x} \quad \Rightarrow \; f'(x) = 2e^{2x} = p \quad \Rightarrow \; x = \frac{1}{2}\ln\frac{p}{2}$$

$$\Rightarrow \; g(p) = \frac{p}{2} - \frac{p}{2}\ln\frac{p}{2}.$$

$$\triangleleft \; g(p) = \frac{p}{2} - \frac{p}{2}\ln\frac{p}{2} \quad \Rightarrow \; g'(p) = -\frac{1}{2}\ln\frac{p}{2} = -x$$

$$\Rightarrow \; p = 2e^{2x} \quad \Rightarrow \; f(x) = e^{2x}.$$

Application to thermodynamics:

- The Legendre transform links different thermodynamic potentials to each other, each with its characteristic set of natural independent variables.
- The partial derivative of one thermodynamic potential with respect to one of its natural independent variables yields the natural independent variable of a different thermodynamic potential.

Application to classical dynamics:

- The Legendre transform links the Lagrangian $L(\{q_i\}, \{\dot{q}_i\})$ and the Hamiltonian $H(\{q_i\}, \{p_i\})$ of a classical mechanical system.
- The generalized velocities $\dot{q}_i$, which naturally appear in the Lagrangian, are replaced in the Hamiltonian by the canonical momenta $p_i \doteq \partial L/\partial \dot{q}_i$ as independent variables.