Useful relations between partial derivatives [tln6]

Consider state variables x, y, z, w. Only two of the variables are independent.

#1 $\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial y}{\partial x}\right)_z^{-1}$

#2 $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$

#3 $\left(\frac{\partial x}{\partial w}\right)_z = \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial w}\right)_z$

#4 $\left(\frac{\partial x}{\partial y}\right)_z = \left(\frac{\partial x}{\partial y}\right)_w + \left(\frac{\partial x}{\partial w}\right)_y \left(\frac{\partial w}{\partial y}\right)_z$

Proof:

Start with total differential of $x(y,z)$: $dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz.$

Substitute total differential of $y(x,z)$: $dy = \left(\frac{\partial y}{\partial x}\right)_z dx + \left(\frac{\partial y}{\partial z}\right)_x dz.$

$$\Rightarrow \left[\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z - 1\right] dx + \left[\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x + \left(\frac{\partial x}{\partial z}\right)_y\right] dz = 0.$$

Expressions in brackets must vanish independently $\Rightarrow$ #1 and #2.

Introduce parameter w: $x(y,z)$ with $y = y(w)$ and $z = z(w)$.

$$dx = \left(\frac{\partial x}{\partial y}\right)_z dy + \left(\frac{\partial x}{\partial z}\right)_y dz \;\Rightarrow\; \frac{dx}{dw} = \left(\frac{\partial x}{\partial y}\right)_z \frac{dy}{dw} + \left(\frac{\partial x}{\partial z}\right)_y \frac{dz}{dw}.$$

Specify path: $z =$const i.e. $dz = 0$: $\Rightarrow$ #3.

Introduce parameter y: $x(y,w)$ with $w = w(y)$ and $z = z(y)$.

$$dx = \left(\frac{\partial x}{\partial y}\right)_w dy + \left(\frac{\partial x}{\partial w}\right)_y dw \;\Rightarrow\; \frac{dx}{dy} = \left(\frac{\partial x}{\partial y}\right)_w + \left(\frac{\partial x}{\partial w}\right)_y \frac{dw}{dy}.$$

Specify path: $z =$const i.e. $dz = 0$: $\Rightarrow$ #4.