[gex96] Inertia tensor from angular momentum of rigid body

Consider a rigid body in purely rotational motion with angular velocity $\boldsymbol{\omega}$ about some instantaneous axis. The (instantaneous) angular momentum can then be constructed from the expression,

$$\mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha), \tag{1}$$

where m_α are infinitesimal mass elements at positions $\mathbf{r}_\alpha$ in a coordinate system fixed to the rigid body with its origin on the axis of rotation. Extract from (1) the expression,

$$I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} r_\alpha^2 - r_{\alpha i} r_{\alpha j} \right], \tag{2}$$

for the inertia tensor by transforming (1) into the form,

$$L_i = \sum_j I_{ij} \omega_j. \tag{3}$$

Solution: