[gex86] Laurent series of analytic functions II

The analytic function,

$$f(z) = \frac{1}{(z+1)(z+3)},$$

has easily recognizable singularities at $z = -1$ and $z = -3$. It can be expanded into three distinct Laurent series at the point $z = 0$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

which converge in the annular regions (i) $|z| < 1$, (ii) $|z| > 3$, and (iii) $1 < |z| < 3$, respectively.
(a) Use the Mathematica commands Series and Apart to calculate the coefficients a_n for $|n| < 5$ of each series.
(b) Use the Cauchy integrals,

$$a_n = \frac{1}{2\pi\imath} \oint_C dw \frac{f(w)}{(w-a)^{n+1}} \quad : \; n = 0, \pm 1, \pm 2, \ldots,$$

to reproduce the results of part (a) with a circular contour parametrized as $w = r\, e^{\imath\phi}$, $a = 0$ with (i) $r = \frac{1}{2}$, (ii) $r = 4$, and (iii) $r = 2$.

Solution: